Boston Public Library
Mathematics and Maps
Title: Similarity

Essential Question: What characteristics make objects the same shape?

Overview:
Although similarity is a mathematical concept that is frequently used in everyday life with models, diagrams and maps, it is also a word used frequently in everyday language, but with a different definition. Two people can be similar because they both have brown hair yet mathematical similarity denotes an actual replica that is a shrunken or larger version of the original. By exploring maps and the idea of similarity, students understand the specifics of the mathematical definition and make sense of the concept by applying their understanding to the real-world.

Grade Range: Grade 6-10 **Time Allocation**: 45 – 60 minutes

Objectives:
1. Students will state the properties of similarity and apply them to determine if shapes are similar.
2. Students will solve similarity problems by applying proportions.
3. Students will define map scale and apply it to solve problems.

Common Core Curriculum Standards:

Grade 6 - Ratios and Proportional Relationships
1. Understand the concept of a ratio: Two quantities are said to be in a ratio of *a* to *b* when for every *a* units of the first quantity there are *b* units of the second. *For example, in a flock of birds, the ratio of wings to beaks might be 2 to 1; this ratio is also written 2:1. In Grade 6, limit to ratios of whole numbers.*
2. Solve for an unknown quantity in a problem involving two equal ratios.

Grade 7 – Geometry - Congruence and similarity
1. Verify experimentally that a dilation with scale factor k preserves lines and angle measure, but takes a line segment of length L to a line segment of length kL.
2. Understand the meaning of similarity: a plane figure is similar to another if the second can be obtained from the first by a similarity transformation (a rigid motion followed by a dilation).

3. Solve problems involving similar figures and scale drawings. *Include computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.*

Note: A dilation is a mathematical transformation that preserves the shape of an object but may change the size.

<u>Grade 8 – Geometry – Congruence and Similarity</u>
Explain using similarity transformations the meaning of similarity for triangles as the equality of all pairs of angles and the proportionality of all pairs of sides.

<u>Grade 9-10 – Geometry – Similiarity</u>
Use triangle similarity criteria to solve problems and to prove relationships in geometric figures

Procedure:

<u>Part 1: Introduction</u>

1. Hand out the Railroad map between Cape Canso and St. Louis (http://maps.bpl.org/id/10501). Have students answer the following questions on their handout.
    1. What is the relationship between the three inset maps?
    2. Are the three inset maps similar? Why or why not?

2. Discuss the student responses. Explain that although the word "similar" is used in a general way, in mathematics the definition is very specific. Lead students to notice that the two bottom maps are more detailed versions of the big map, with the map in the bottom right corner the most detailed. Discuss what this means with the class. In other words, what does it mean in terms of the distances represented in Massachusetts? What does it mean in terms of the size of Massachusetts? What does it mean in terms of the shapes of Massachusetts?

<u>Part 2: Similarity</u>

1. Give a basic definition of similarity: "Two objects are similar if they are the same shape but not the same size." Discuss, "What makes objects the same shape – what characteristics go into understanding a shape?" Students should see that the "shape" can be determined by specific mathematics criteria: the angles and the side lengths.

2. Have students outline the state of Massachusetts in the top and bottom left maps. How do we know that these are the same shape? Model with students how to measure angles and distances on each map. They should then complete the data chart and draw conclusions.

Ultimately, students should find that in similar shapes, angles are preserved but sides are proportional.

Note: When measuring actual distances, accuracy is important. The ratios will not be exactly equal, but should be around 4. Ratios that are very different should be re-checked. This can also be a good time to discuss accuracy and possible reasons for errors.

3. Define the two properties of similarity. Have students copy the definition in their handout.

   Two shapes are similar if
   a. Corresponding angles are congruent.
   b. Corresponding sides are proportional.

4. Discuss the idea of proportionality and what the "ratio of corresponding sides" means. Do an example as described below.
   "Corresponding sides are proportional" means that if you take the ratio of corresponding sides on the two maps of Massachusetts, the ratios will be the same no matter what sides you use. Because of this, you can set up a proportion to find missing sides lengths.

   For example, the top to bottom distance across Massachusetts is approximately 1.375 inches on the larger map. To find what that distance is on the smaller map, show the students how to set up the ratio:

   <u>Larger Area Map Distance</u>
   Smaller Area Map Distance

   $$= \frac{1.375\,inches}{x\,inches}$$

   Using the ratio, create a proportion to solve for the missing side length. To do this, use another ratio of corresponding sides where the distances on both the smaller and larger maps are known. Any other ratio can be used as long as both the numerator and denominator can be filled in. For example, the length of the left side of Massachusetts can be used. The left side on the larger map is approximately 1.3125 inches while the left side on the smaller map is approximately .3125 inches. We can now set up our proportion.

   <u>Larger Area Map Distance</u>
   Smaller Area Map Distance

   $$= \frac{1.375\,inches}{x\,inches} = \frac{1.3125\,inches}{.3125\,inches}$$

   Solve by cross-multiplying.

$$\frac{1.375\,inches}{x\,inches} = \frac{1.3125\,inches}{.3125\,inches}$$
$$1.3125 \times x = 1.375 \times .3125$$
$$1.3125x = .43$$

Divide both sides by 1.3125 to solve for x, the missing side length.

$$\frac{1.3125x}{1.3125} = \frac{.43}{1.3125}$$

$$x = .33\,inches$$

Thus, the distance from top to bottom on the smaller map of Massachusetts is .33 inches.

5. Using this information, have students complete the table, calculating missing distances from the maps using proportions. Model the first one with students.

Part 3: Extension – Determining Scaling Factors

1. Note that the large area map and the smaller area map in the right hand corner both have scales. Have the students measure the scale bars and write down the scales. The students should then simplify the scales so they are in the form 1 inch = ________ . An approximation for the scales and the work is shown below.

   Scale for Larger Area Map:

   Map Distance
   Actual Distance

   $$= \frac{1.25\,inches}{100\,miles}\frac{\div 1.25}{\div 1.25} = \frac{1\,inch}{80\,miles}$$

   Scale for Smaller Area Map:

   Map Distance
   Actual Distance

   $$= \frac{1.25\,inches}{10{,}000\ feet}\frac{\div 1.25}{\div 1.25} = \frac{1\,inch}{8000\ feet}$$

2. These scales indicate the relationship between distances on the map and distances in the actual world. But what do theses scales tell us about how much smaller the maps are than

the actual world? To do this, we need to find the scaling factors. The scaling factor is the number of times smaller the map is than the real world. Have students define this on their handout.

3. To find the scaling factors, we need to convert our equation scales to a representative fraction. This is a ratio that represents the relationship between map distance and the actual world where the units are the same and are therefore cancelled out. So given our scale $\frac{1\,inch}{80\,miles}$, to convert this to a representative fraction, we need to convert 80 miles to inches. We first convert 80 miles to feet. Have students take notes on the process for converting the equation scale to a ratio scale to get the scaling factor. Since we currently have 80 miles and we know that $\frac{1\,mile}{5280\,feet}$, we can set up ratios to convert from miles to feet.

$$\frac{80\,miles}{1} \times \frac{5{,}280\ feet}{1\,mile} = \frac{422{,}400\,miles \cdot\ feet}{1\,mile} = 422{,}400\ feet$$

Note that the unit of miles cancels and the remaining unit is “feet.”

4. So far, we have the following for our scale:

$$Scale = \frac{Distance\ on\ Map}{Distance\ in\ Actual\ World} = \frac{1\ inch}{80\,miles} = \frac{1\ inch}{422{,}400\ feet}$$

5. Now, if we can convert the 422,400 feet into inches, we’ll be able to cancel the inches to have our ratio scale and our scaling factor. Since we currently have 422,400 feet and we know that $\frac{1\ foot}{12\,inches}$, we can set up ratios to convert from feet to inches.

$$\frac{422{,}400\ feet}{1}\ x\ \frac{12\,inches}{1\ foot} = \frac{5{,}068{,}800\ feet\ \cdot inches}{1\ foot} = 5{,}068{,}800\,inches$$

Thus, we have $\frac{1\ inch}{5{,}068{,}800\,inches}$. Canceling the unit of inches in the numerator and denominator, we get the ratio scale of $\frac{1}{5{,}068{,}800}$.

6. This means the following:

   The larger area map is $\frac{1}{5{,}068{,}800}$ or .000000197 times smaller than the actual world.
   The actual world is 5,068,800 times larger than the map.

   And, the scaling factor for the map is $\frac{1}{5{,}068{,}800}$ or .0000000197.

7. Have the students repeat the process to determine the scaling factor for the map on the bottom left corner of the page and answer the questions on their handout.

Part 6: Assessment

1. Have students complete the assignment on similarity using the Railroad map between Cape Canso and St. Louis and the United States railway map.

2. Collect the assignments and review answers.

Materials Needed:

- Maps Used:
  - Railroad map between Cape Canso and St. Louis (http://maps.bpl.org/id/10501)
  - United States Railway Map (http://maps.bpl.org/id/15602)

  (Note: Maps are replicated at the end of the handout – they would be easier to use if printed in landscape form instead of portrait.)
- Markers (for outlining states)
- Scissors
- Pencils
- Rulers
- Protractor

Similarity

Name ____________

Date _____________

Part 1: Introduction

1. What is the relationship between the three inset maps?
____
____
____

2. Are the three inset maps similar? Why or why not?
____
____
____

Part 2: Similarity

In general, two objects are similar if they are the same ________________, but not the same ______________.

1. Outline the state of Massachusetts in the maps below.

2. Follow along with your teacher for how to measure angles and distances. Then, complete the chart to investigate patterns in the angles of the two versions of the state of Massachusetts.

Angle	**Angle Measurement on Smaller Map**	**Angle Measurement on Larger Map**
Top left angle in Massachusetts		
Bottom left angle in Massachusetts		
Angle adjacent to Rhode Island		
Choose one more angle		

What conclusions can you draw about the angles of similar shapes?
____
____
____

3. Complete the chart to investigate patterns in the distances in the two versions of the state of Massachusetts.

Distance	**Distance on Smaller Map**	**Distance on Larger Map**	**How much larger? Ratio of** <u>*Larger Distance*</u> *Smaller Distance*
Left Side of Massachusetts			
Top Side of Massachusetts			
Bottom Side of Massachusetts			
Choose one more distance.			

What conclusions can you draw about the distances in similar shapes?
____
____

4. The formal definition of similarity is based on two properties. The mathematical definition of similarity is as follows:

   Two shapes are similar if

   (1.) ____

   (2.) ____

   Notes/Example:

5. Knowing that the two maps of Massachusetts are similar, complete the table below. Show your proportion work where needed.

Distance/Angle	Smaller Map	Larger Map
Distance from Boston to Pittsfield		2.9375 inches
Boston Harbor Angle	92°	
Diagonal Length of Massachusetts	1.1875 inches	

Part 3: Extension – Determining Scaling Factors

1. Note that the big map and the smaller map in the right hand corner both have scales. Measure the graphical scales and write them down below in ratio form. Then simplify the scales so the actual distance is what is represented by 1 inch on the map.

   Scale for the Larger Area Map:

   Scale for the Smaller Area Map:

Vocabulary	Definition
Scaling Factor	

2. To find the scaling factors, we need to convert our equation scales to a representative fraction. Follow along with your teacher to convert the equation scale into a representative fraction.

Representative Fraction for Smaller Map: ________________________________

This means:

- Distances in the actual world are _______________ times larger than _______________ .
- Distances on the map are __________________times smaller than _________________ .

Scaling Factor for the Bigger Map: ___________________________

3. Repeat the process to find the representative fraction and scaling factor for the smaller area map.

Representative Fraction for the Larger Area Map: ______________________________

This means:

- Distances in the actual world are ______________ times larger than ______________ .
- Distances on the map are _________________times smaller than ________________ .

Scaling Factor for the Bigger Map: _________________________

4. What is the relationship between the bigger map and the smaller map? In other words, what is the scaling factor from the bigger area map to the smaller area map? Show your work.

The smaller area map is _____________ times larger than the bigger area map.

Part 4: Assessment

1. Compare the two maps, “Railroads between Cape Canso and St. Louis” and “Official Guide to Railways of the United States.”
    a. What observations do you make about the two maps?
    ____
    ____
    ____

    b. Are the states that are in both maps similar in shape? Explain.
    ____
    ____
    ____
    ____

2. Choose three corresponding angles on the maps and compare their measurements.

Angle	**Angle Measurement on United States map**	**Angle Measurement on Cape Canso map**

3. Choose three pairs of corresponding sides on the maps and compare their ratios.

Distance	**Distance on United States map**	**Distance on Cape Canso map**	**How much larger? Ratio of** *U.S. Distance* / *Cape Canso Distance*

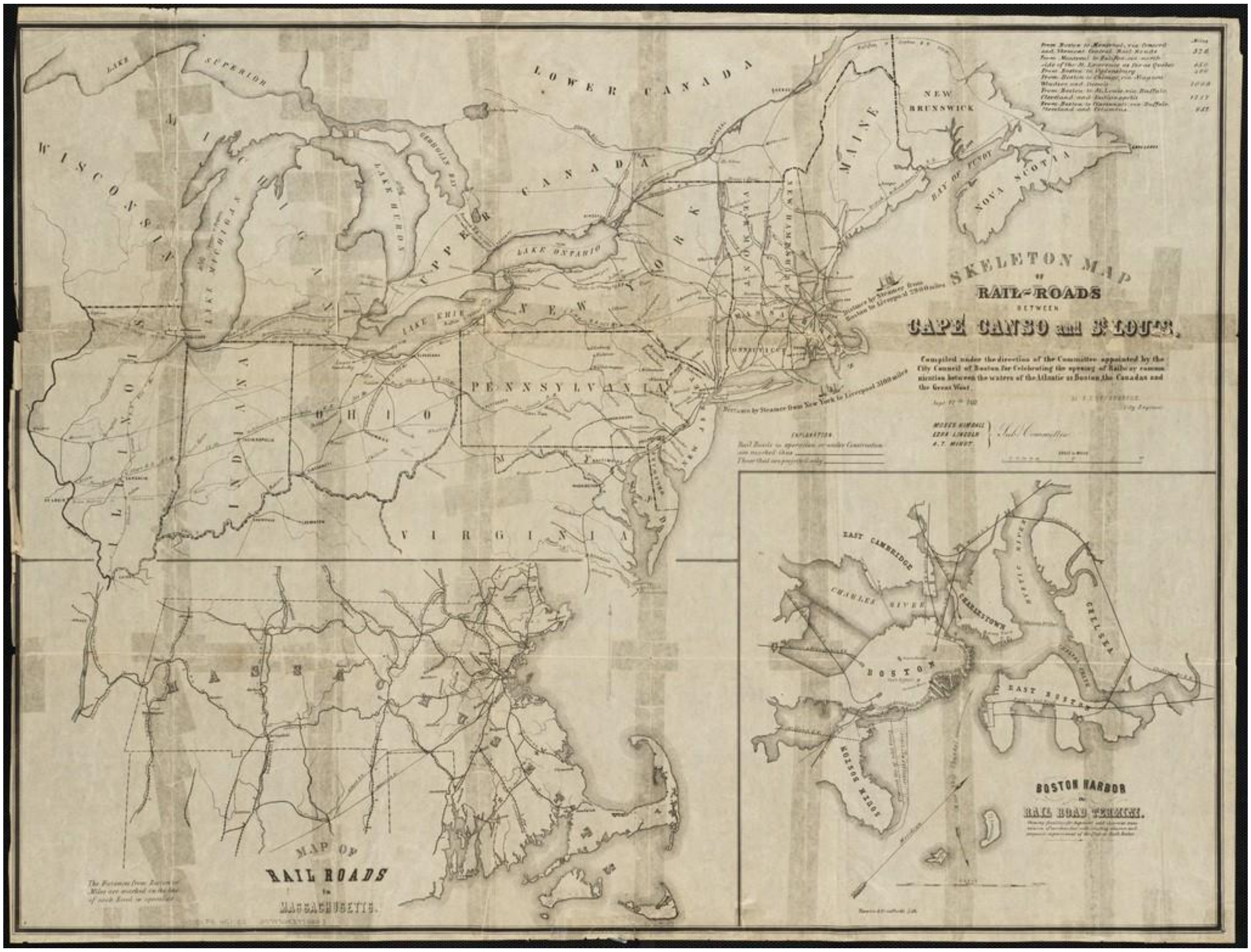
4. Why are objects in these two maps similar? Justify your response using the data you collected and the properties of similarity.

____
____
____
____
____
____
____

5. Knowing that objects in theses two maps are similar, complete the table below. Show your proportion work where needed.

Distance/Angle	United States Map	Cape Canso Map
Width of Pennsylvania		1 inch
Angle at top right corner of Indiana	90°	
Distance between Boston and the southern most tip of Lake Michigan	2.875 inches	

6. Find the scaling factor for the United States map. In other words, determine how many times larger distances in the real world are than the map. The equation scale is .625 inches = 200 miles.



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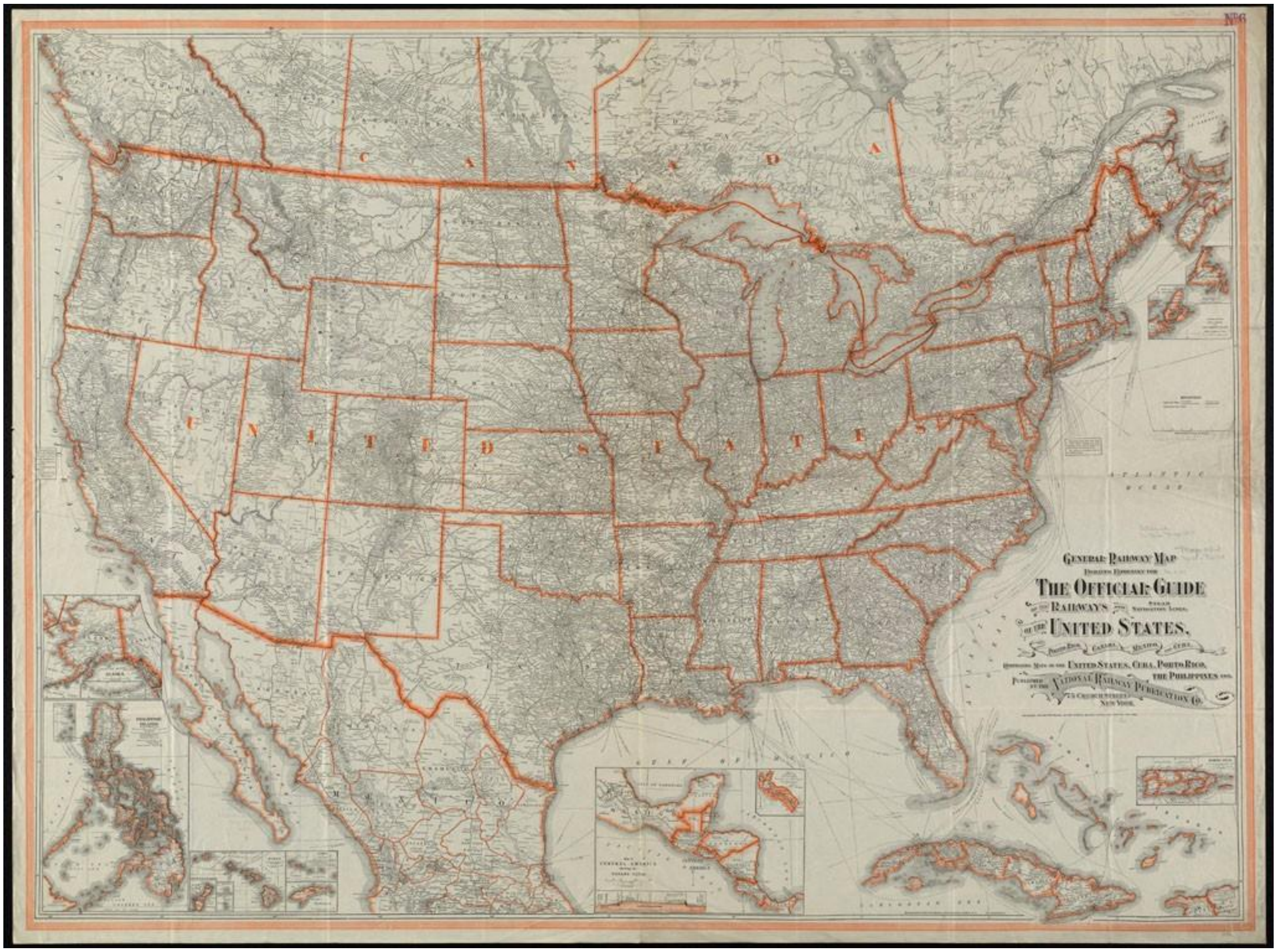
http://www.leventhalmap.org/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